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GENERATION OF TURBULENCE IN COUETTE FLOW BETWEEN EXCENTRIC CYLINDERS

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1

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Generation of Turbulence in Couette Flow between Excentric Cylinders.

1.) Introductory

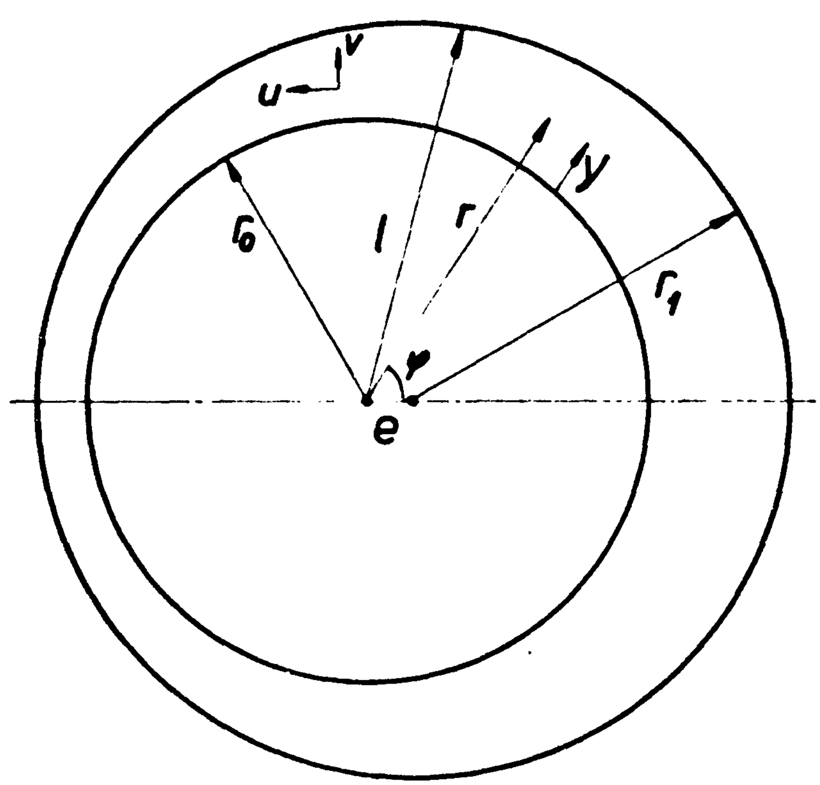
Couette flow is besides the Hagen-Poiseuille flow the fundamental experiment for the study of the flow properties of liquids and the study of stability and transition to turbulence in ducts. Although the theory of stability was most successful by discovering the generation of cellular vortices [1] when the centrifugal forces act stabilizing there does not exist a complete understanding of the transition phenomenon to turbulence which was observed [1,2,3,4] when the centrifugal forces act stabilizing. An example for the first case is a rotating inner cylinder and the outer cylinder at rest and for the second case the outer cylinder rotating and the inner cylinder at rest. Only recently it was pointed out [5] that the observed transition could have been caused by vibrations or excentricities which were produced by imperfections of the Couette apparatus used in the experiments. The use of ball bearings, large ratios of the length to the diameter of the cylinders, large dimensions and cylinders bent from sheet point to the likelihood of such imperfections. When annihilating excentricities and vibrations it indeed was found [5] that the flow is completely stable up to Reynolds numbers jet not attained. Also the theoretical stability proof on the basis of small two-dimensional perturbations shows stability [5,6]. Furthermore with determined excentricities transition to turbulence was obtained with a definite dependency on the Reynolds number. There seem to be enough indications for the creation of turbulence by excentricities. It is the purpose of this investigation to give a theoretical explanation of this transition phenomenon in the presence of excentric cylinders.

Transition can be caused either by flow instability or by separation. In the latter case there occurs near the wall counter flow wich initiates transition. Preliminary experimental observations gave strong evidence for this type of transition. Therefore the investigation will deal with the separation effect opposite to the original intention.

2.) General assumptions and notations.

As mentioned before a rotating outer cylinder and the inner cylinder at rest will be assumed. Plane motion will be regarded. This means that this investigation refers to relatively long cylinders so that end effects will not influence the middle part of the flow.

The center of the inner cylinder will be regarded as center of reference (fig. 1). The excentricity of the outer cylinder of radius r_1 will be denoted by e. The ratio e/r_1 is regarded as small so that higher orders of this ratio can be neglected. The discussion of the boundary conditions will show



that the satisfaction of these conditions at the excentric boundary involves rather cumbersome numerical calculations. Therefore the outer boundary will be altered some what in the way that the boundary conditions will be shifted from the excentric to a hypothetic centric circular boundary with the same radius as the excentric boundary. In the case of a small ratio of the excentricity to the mean width of the gap this is of negligible influence whereas at larger excentricities the boundary conditions give a periodically alternating in- and outflow at the hypothetical boundary. One may say that these conditions are still in the neighborhood of the real boundary conditions if the excentricity is a larger fraction of the average width of the gap.

Navier-Stokes equations will be considered with out neglecting any terms. However the inertia terms will be linearized by assuming as mean flow the Couette flow between centric cylinders. Therefore only small perturbations should be generated by the excentricity. This means that the investigation is restricted to excentricities which are small in comparison to the mean width of the gap.

The linearization is not in agreement with the physical problem of separation. Indeed separation is an effect of finite inertia forces as fluid particles are subjected to a finite deceleration. Therefore the calculations presented here can only show the tendency to separation. One can not expect that the calculated values of the parameter which characterizes separation will be in good agreement with experimental values. The comparison with own experiments will show that the calculated values are to small. But this lack in agreement characterizes all calculations of similar problems as the mathematical difficulties are invincible with out the simplification of linearization.

3.) Boundary conditions.

The circumferential velocity of the outer cylinder will be denoted by U^*. With the center of reference in the center of the inner cylinder one has according to fig. 2 the tangen-

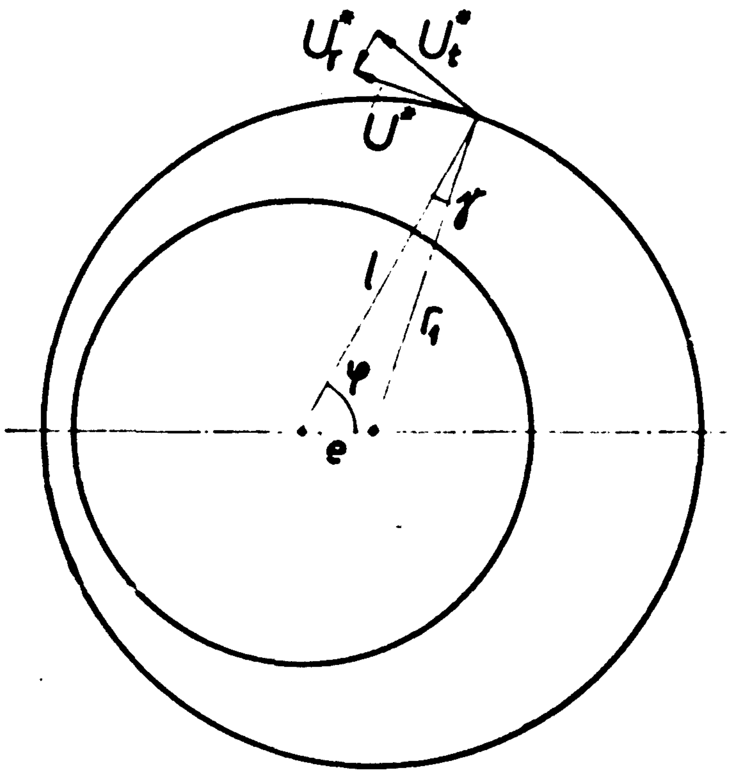


Fig. 2

tial and radial components

$$U^*_t = U^* \cos\gamma \quad , \quad U^*_r = - U^* \sin\gamma \tag{1}$$

Introducing the geometric relations

$$r_1 \sin\gamma = e \sin\varphi \quad , \quad r_1 \cos\gamma + e \cos\varphi = l \tag{2}$$

one obtains

$$U_t^* = U^* \frac{r_0}{r_1}\left(\frac{l}{r_0} - \frac{e}{r_0}\cos\varphi\right), \quad U_r^* = - U^* \frac{r_0}{r_1}\,\frac{e}{r_0}\sin\varphi \tag{3}$$

From (2) one derives

$$l = e\cos\varphi + r_1\sqrt{1 - \frac{e^2}{r_1^2}\sin^2\varphi}$$

The series expansion is, if the dimensionless excentricity

$$\varepsilon = \frac{e}{r_0} \tag{4}$$

is introduced

$$\frac{\ell}{r_0} = \varepsilon \cos\varphi + \frac{r_1}{r_0}\left[1 - \frac{1}{4}\frac{r_0^2}{r_1^2}\,\varepsilon^2\left(1-\cos 2\varphi\right)\right] \tag{5}$$

With the above mentioned assumption

$$\varepsilon^2 \ll 1 \tag{6}$$

and the notations

$$\sigma = \frac{\ell}{r_0}\ , \quad \varrho = \frac{r_1}{r_0}\ , \quad \lambda = \varepsilon/\varrho \tag{7a}$$

one obtains

$$\sigma = \varrho\,(1 + \lambda\cos\varphi) \tag{7b}$$

This gives when introduced in (3) the following expressions for the tangential and radial components of the circumferential velocity U^* of the outer cylinder

$$U_t^* = U^*\ , \quad U_r^* = -U^*\lambda\sin\varphi \tag{8}$$

Introducing the mean width

$$h_0 = r_1 - r_0 \tag{9}$$

and

$$\delta_0 = \frac{h_0}{r_0} = \varrho - 1 \tag{10}$$

the second equation (8) can also be written

$$U_r^* = -U^*\frac{\varepsilon}{1+\delta_0}\sin\varphi \tag{11}$$

This shows that excentricities of the order $\varepsilon^2 \ll 1$ influence only the radial velocity component.

The width of the gap is

$$h = r_1 - r_0 + e\cos\varphi \tag{12}$$

Then

$$h = h_0 \left(1 + \frac{e}{h_0} \cos\varphi\right) \tag{13}$$

Later the variable

$$y = \frac{r - r_0}{r_0} \tag{14}$$

will be introduced. It is the dimensionless distance from the inner wall. The value of y at the outer boundary is δ according to (1o). (13) gives

$$\delta = \frac{h_0}{r_0} \left(1 + \frac{e}{h_0} \cos\varphi\right) \tag{15}$$

With the notation

$$\frac{e}{h_0} = \alpha \tag{16}$$

one obtains

$$\delta = \delta_0 (1 + \alpha \cos\varphi) \tag{17}$$

As the calculations will be restricted to

$$\alpha = \frac{e}{h_0} \ll 1 \tag{18}$$

the higher powers of δ can be approximated by

$$\delta^2 = \delta_0^2 (1 + 2\alpha \cos\varphi) \, , \qquad \delta^3 = \delta_0^3 (1 + 3\alpha \cos\varphi) \, , \quad \ldots \tag{19}$$

It will be shown that the limitation (18) is not absolutely necessary. Nevertheless it will be introduced as otherwise the calculations get to extensive.

4.) Basic equations.

Denoting by u the tangential by v the radial velocity (fig.1) by p pressure, by ν kinematic viscosity, by index r, φ differentiations with respect to r, φ the Navier-Stokes equations for the circumferential and radial direction refering to the twodimensional motion are

$$rvu_r + uu_\varphi + uv = -\frac{1}{\varrho} p_\varphi + \nu \left[u_{rr} + \frac{u_r}{r} - \frac{u}{r^2} + \frac{u_{\varphi\varphi}}{r^2} + \frac{2}{r^2} v_\varphi \right] r \tag{2o}$$

$$vv_r + \frac{u}{r} v_\varphi - \frac{u^2}{r} = -\frac{1}{\varrho} p_r + \nu \left[v_{rr} + \frac{v_r}{r} - \frac{v}{r^2} + \frac{v_{\varphi\varphi}}{r^2} - \frac{2}{r^2} u_\varphi \right] \tag{21}$$

Introducing the stream function ψ defined by

$$u = -\psi_r \quad , \qquad v = \frac{1}{r}\psi_\varphi$$

and differentiating (2o) with respect to r, (21) with respect to φ then eliminating the pressure and introducing Laplaces operator

$$\Delta\psi = \psi_{rr} + \frac{1}{r}\psi_r + \frac{1}{r^2}\psi_{\varphi\varphi}$$

one finally obtains

$$\psi_\varphi \Delta\psi_r - \psi_r \Delta\psi_\varphi = \nu\, r\, \Delta\Delta\psi \qquad (22$$

The stream function will be composed of two parts ψ^* of the mean flow and ψ' of the perturbation

$$\psi = \psi^* + \psi'$$

As mentioned before for the mean flow the Couette flow between centric cylinders will be introduced the velocities of which are

$$\psi_r^* = -U = -\frac{U^* r_1}{\left(\frac{r_1}{r_0}\right)^2 - 1}\left(\frac{r}{r_0^2} - \frac{1}{r}\right) \quad , \qquad \frac{1}{r}\psi_\varphi^* = V = 0 \qquad (23$$

With this (22) is transformed to

$$U\Delta\psi_\varphi' + \psi_\varphi'\Delta\psi_r' - \psi_r'\Delta\psi_\varphi' = \nu r \Delta\Delta\psi' \qquad (24$$

Linearizing this equation by neglecting the second order terms in ψ' writing ψ for ψ' and introducing the abbreviation

$$K = \frac{U^* r_1}{\left[\left(\frac{r_1}{r_0}\right)^2 - 1\right]\nu} \qquad (25$$

one obtains

$$K\left(\frac{r^2}{r_0^2} - 1\right)\Delta\psi_\varphi = r^2\Delta\Delta\psi \qquad (26$$

The variable y defined by (14) will be introduced. Then the equation

$$K(2y+y^2)\left[(1+y)^2\psi_{yy\varphi} + (1+y)\psi_{y\varphi} + \psi_{\varphi\varphi\varphi}\right] = \left[(1+y)^4\psi_{yyyy} + 2(1+y)^3\psi_{yyy} - \right.$$
$$\left. -(1+y)^2\psi_{yy} + (1+y)\psi_y + 4\psi_{\varphi\varphi} - 2(1+y)\psi_{\varphi\varphi y} + 2(1+y)^2\psi_{\varphi\varphi yy} + \psi_{\varphi\varphi\varphi\varphi}\right] \qquad (2.$$

is obtained

5.) Solution for zero inertia terms.

According to the geometry of the problem a solution periodic in the circumferential angle φ (fig. 1) is to be expected. Solutions with this property are well known for vanishing inertia terms as the basic equations then are reduced to the biharmonic differential equations. One has with the dimensionless radius $r = r/r_0$

$$\begin{aligned}\psi = a_1 + a_2 \ln r + a_3 r^2 &+ \cos\varphi\,(b_1 r + b_2 r^{-1} + b_3 r^3 + b_4 r \ln r)\\ &+ \cos 2\varphi\,(c_1 r^2 + c_2 r^{-2} + c_3 r^4 + c_4 \qquad)\\ &+ \cos 3\varphi\,(d_1 r^3 + d_2 r^{-3} + d_3 r^5 + d_4 r^{-1})\\ &+ \cdots\end{aligned}$$

From this one obtains the velocity components

$$\begin{aligned}u &= -\psi_r\\ &= -\{a_2 r^{-1} + 2a_3 r + \cos\varphi\,(b_1 - b_2 r^{-2} + 3b_3 r^2 + b_4 + b_4 \ln r)\\ &\qquad + \cos 2\varphi\,(c_1 2r - 2c_2 r^{-3} + 4c_3 r^3)\\ &\qquad + \cos 3\varphi\,(3d_1 r^2 - 3d_2 r^{-4} + 5d_3 r^4 - d_4 r^{-2})\\ &\qquad + \cdots\}\end{aligned} \tag{28}$$

$$\begin{aligned}v = \frac{1}{r}\psi_\varphi = -\{ &\sin\varphi\,(b_1 + b_2 r^{-2} + b_3 r^2 + b_4 \ln r)\\ &+ 2\sin 2\varphi\,(c_1 r + c_2 r^{-3} + c_3 r^3 + c_4 r^{-1})\\ &+ 3\sin 3\varphi\,(d_1 r^2 + d_2 r^{-4} + d_3 r^4 + d_4 r^{-2})\\ &+ \cdots\}\end{aligned} \tag{29}$$

The boundary conditions $u = v = 0$ at $r = 1$ give

$$
\begin{aligned}
&b_1 + b_2 + b_3 = 0 && a_2 + 2a_3 = 0 \\
&c_1 + c_2 + c_3 + c_4 = 0 && b_1 - b_2 + 3b_3 + b_4 = 0 \\
&\dots && 2c_1 - 2c_2 + 4c_3 = 0 \\
& && \dots
\end{aligned}
\tag{30}
$$

Introducing (7,8) in (28,29) one obtains for the outer boundary, when powers of σ are developed into power series of λ and only first order terms in λ are regarded

$$
\begin{aligned}
-U^* = {} & a_2 \varrho^{-1}(1 - \lambda\cos\varphi) + 2a_3\varrho(1 + \lambda\cos\varphi) \\
& + \cos\varphi\,[b_1 - b_2\varrho^{-2}(1 - 2\lambda\cos\varphi) + 3b_3\varrho^2(1 + 2\lambda\cos\varphi) \\
& \qquad + b_4(1 + \ln\varrho) + b_4\ln(1 + \lambda\cos\varphi)] \\
& + \cos 2\varphi\,[2c_1\varrho(1 + \lambda\cos\varphi) - 2c_2\varrho^{-3}(1 - 3\lambda\cos\varphi) \\
& \qquad + 4c_3\varrho^3(1 + 3\lambda\cos\varphi)] \\
& \dots
\end{aligned}
$$

$$
\begin{aligned}
U^*\lambda\sin\varphi = {} & \sin\varphi\,[b_1 + b_2\varrho^{-2}(1 - 2\lambda\cos\varphi) + b_3\varrho^2(1 + 2\lambda\cos\varphi) + b_4\ln\varrho \\
& \qquad + b_4\ln(1 + \lambda\cos\varphi)] \\
& + 2\sin 2\varphi\,[c_1\varrho(1 + \lambda\cos\varphi) + c_2\varrho^{-3}(1 - 3\lambda\cos\varphi) \\
& \qquad + c_3\varrho^3(1 + 3\lambda\cos\varphi) + c_4\varrho^{-1}(1 - \lambda\cos\varphi)] \\
& \dots
\end{aligned}
$$

Rearranging terms with respect to multiples of φ one obtains

$$
\begin{aligned}
-U^* = {} & a_2\varrho^{-1} + 2a_3\varrho + b_2\varrho^{-2}\lambda + 3b_3\varrho^2\lambda + \tfrac{1}{2}b_4\lambda \\
& + \cos\varphi\,[-a_2\varrho^{-1}\lambda + 2a_3\varrho\lambda + b_1 - b_2\varrho^{-2} + 3b_3\varrho^2 + b_4(1 + \ln\varrho) \\
& \qquad + c_1\varrho\lambda + 3c_2\varrho^{-3}\lambda + 6c_3\varrho^3\lambda] \\
& + \cos 2\varphi\,[b_2\varrho^{-2}\lambda + 3b_3\varrho^2\lambda + \tfrac{1}{2}\lambda b_4 + 2c_1\varrho - 2c_2\varrho^{-3} + 4c_3\varrho^3 + \cdots\ ] \\
& \dots
\end{aligned}
$$

$$
\begin{aligned}
U^*\lambda\sin\varphi = {} & \sin\varphi\,[b_1 + b_2\varrho^{-2} + b_3\varrho^2 + b_4\ln\varrho + c_1\varrho\lambda - 3c_2\varrho^{-3}\lambda + 3c_3\lambda\varrho^3 - c_4\lambda\varrho^{-1}] \\
& + \sin 2\varphi\,[-b_2\varrho^{-2}\lambda + b_3\varrho^2\lambda + \tfrac{1}{2}b_4\lambda + 2c_1\varrho + 2c_2\varrho^{-3} + 2c_3\varrho^3 + 2c_4\varrho^{-1} + \dots] \\
& \dots
\end{aligned}
$$

Equalizing terms with the same multiples in φ on the right and left side one obtains 5 equations which together with the 5 equations (30) for the inner boundary determine the 10 constants a_2, a_3, b_1, b_2, b_3, b_4, c_1, c_2, c_3, c_4.

The numerical evaluation of the constants is given in table I for $\lambda = 0{,}08$. Terms including 6φ are considered to show the convergence. One sees that it is sufficient to consider coefficients up to d that means up to the terms containing 3φ.

This solution surely would not be sufficient to show the separation effect. It merely should demonstrate the satisfaction of the conditions at an excentric boundary. However this solution is part of the solution which considers the linearized inertia terms. This will be shown later.

Table I

$a_2 = $ [illegible]$,792011233$	$d_1 = 0,381917028$	$f_1 = -0,019382031$
$a_3 = -0,896005614$	$d_2 = 0,167656633$	$f_2 = -0,036423168$
$b_1 = -2,103841400$	$d_3 = -0,198725809$	$f_3 = 0,008220991$
$b_2 = 7,431828035$	$d_4 = -0,350847855$	$f_4 = 0,047584208$
$b_3 = -5,327986714$	$e_1 = -0,023912624$	$g_1 = 0,011262653$
$b_4 = 25,519620761$	$e_2 = 0,022420572$	$g_2 = 0,018200651$
$c_1 = -3,047627428$	$e_3 = 0,023539611$	$g_3 = -0,006352103$
$c_2 = -1,135904752$	$e_4 = -0,022047551$	$g_4 = -0,023111201$
$c_3 = 0,955861333$		
$c_4 = 3,227670052$		

6.) Solution considering linearized inertia terms.

Similar to the solution for zero inertia terms the expression

$$\psi = \sum_{0}^{s} f_s(y) \cos s\varphi + \sum_{1}^{s} g_s(y) \sin s\varphi \tag{31}$$

will be introduced with the power series

$$f_s(y) = \sum_{i=1}^{\infty} a_{si}\, y^i \quad , \quad g_s(y) = \sum_{i=1}^{\infty} m_{si}\, y^i \tag{32}$$

The coordinate y is defined by (14). Introducing this expression into the basic equation (27), then putting the terms of $\cos s\varphi$, $\sin s\varphi$ to zero and comparing equal powers of y one obtains equations with which the coefficients of each series can be expressed by the first four ones. To satisfy the conditions at the outer boundary one has to introduce the expression (17, 19) for y and its powers. Powers and products of $\cos s\varphi$, $\sin s\varphi$ will occur which can be expressed by the sinus and cosinus of multiples of the angle φ as shown in section 5 for zero inertia terms. Then each s- term in (32) would demand the determination of four constants from the boundary conditions.

These evaluations indeed would be exceedingly cumbersome. Therefore the limitation (18) will be introduced so that the excentric boundary may be replaced by a centric boundary with radius r_1. Now instead of (31) the expression

$$\psi = e\left[f(y)\cos\varphi + g(y)\sin\varphi\right] \tag{33}$$

with

$$f = \sum_{0}^{\infty} a_i y^i \quad , \quad g = \sum_{0}^{\infty} m_i y^i \tag{34}$$

is sufficient.

The following procedure is similar to the one mentioned before: Putting the terms of $\cos\varphi$ and of $\sin\varphi$ to zero and comparing equal powers of y one obtains equations which allow to express the series coefficients by the first four ones of each series. This is a similar method used by Görtler to treat the free boundary layer flow near a corrugated plate [7] .

The following expressions are obtained for the various coefficients a, m. The laborious derivation will be omitted here.

Following this way one first obtains by introducing (33) in (27) the differential equation

$$
\begin{aligned}
&\{\cos\varphi\,(-f^{IV} - 2f''' + 3f'' - 3f' + 3f) \\
&\quad + \sin\varphi\,(-g^{IV} - 2g''' + 3g'' - 3g' + 3g)\} \\
&+ y\,\{\cos\varphi\,[-4f^{IV} - 6f''' + 6f'' - 3f' + K(-2g + 2g' - 2g'')] \\
&\quad + \sin\varphi\,[-4g^{IV} - 6g''' + 6g'' - 3g' + K(2f - 2f' + 2f'')]\} \\
&+ y^2\,\{\cos\varphi\,[-6f^{IV} - 6f''' + 3f'' + K(-g + 3g' + 5g'')] \\
&\quad + \sin\varphi\,[-6g^{IV} - 6g''' + 3g'' + K(f - 3f' - 5f'')]\} \\
&+ y^3\,\{\cos\varphi\,[-4f^{IV} - 2f''' + K(g' + 4g'')] \\
&\quad + \sin\varphi\,[-4g^{IV} - 2g''' + K(-f' - 4f'')]\} \\
&+ y^4\,\{\cos\varphi\,[-f^{IV} + Kg''] + \sin\varphi\,[-g^{IV} - Kf'']\} = 0
\end{aligned}
$$

Then introducing the series expansions (34) for f,g and expressing the series coefficients by the first four ones of each series the following expressions are obtained, when the conditions $u = v = 0$ at the inner boundary are introduced (See (44)).

$$\frac{a_4}{m_4} = -0{,}5\,\frac{a_3}{m_3} + 0{,}25\,\frac{a_2}{m_2} \qquad (35$$

$$\frac{a_5}{m_5} = 0{,}45\,\frac{a_3}{m_3} - 0{,}25\,\frac{a_2}{m_2} + K\left(\pm\, 0{,}03\,\frac{m_2}{a_2}\right) \qquad (36$$

$$\frac{a_6}{m_6} = -0{,}425\,\frac{a_3}{m_3} + 0{,}25\,\frac{a_2}{m_2} + K\left(\pm\, 0{,}03\,\frac{m_3}{a_3} \mp 0{,}016\,\frac{m_2}{a_2}\right) \qquad (37$$

$$\genfrac{}{}{0pt}{}{a_7}{m_7} = 0{,}410714 \genfrac{}{}{0pt}{}{a_3}{m_3} - 0{,}25 \genfrac{}{}{0pt}{}{a_2}{m_2} + K(\mp 0{,}0380952 \genfrac{}{}{0pt}{}{m_3}{a_3} \pm 0{,}0142857 \genfrac{}{}{0pt}{}{m_2}{a_2}) \qquad (38$$

$$\genfrac{}{}{0pt}{}{a_8}{m_8} = -0{,}4017857 \genfrac{}{}{0pt}{}{a_3}{m_3} + 0{,}25 \genfrac{}{}{0pt}{}{a_2}{m_2} + K(\pm 0{,}0392857 \genfrac{}{}{0pt}{}{m_3}{a_3} \mp 0{,}0130952 \genfrac{}{}{0pt}{}{m_2}{a_2})$$

$$- K^2 \cdot 0{,}00079365 \genfrac{}{}{0pt}{}{a_2}{m_2} \qquad (39$$

$$\genfrac{}{}{0pt}{}{a_9}{m_9} = 0{,}39583 \genfrac{}{}{0pt}{}{a_3}{m_3} - 0{,}25 \genfrac{}{}{0pt}{}{a_2}{m_2} + K(\mp 0{,}0392857 \genfrac{}{}{0pt}{}{m_3}{a_3} \pm 0{,}01240079 \genfrac{}{}{0pt}{}{m_2}{a_2}$$

$$+ K^2(-0{,}000661375 \genfrac{}{}{0pt}{}{a_3}{m_3} + 0{,}0010582 \genfrac{}{}{0pt}{}{a_2}{m_2}) \qquad (40$$

$$\genfrac{}{}{0pt}{}{a_{10}}{m_{10}} = -0{,}3916 \genfrac{}{}{0pt}{}{a_3}{m_3} + 0{,}25 \genfrac{}{}{0pt}{}{a_2}{m_2} + K(\pm 0{,}0388690 \genfrac{}{}{0pt}{}{m_3}{a_3} \mp 0{,}0119544 \genfrac{}{}{0pt}{}{m_2}{a_2})$$

$$+ K^2(0{,}00128307 \genfrac{}{}{0pt}{}{a_3}{m_3} - 0{,}001190476 \genfrac{}{}{0pt}{}{a_2}{m_2}) \qquad (41$$

$$\genfrac{}{}{0pt}{}{a_{11}}{m_{11}} = 0{,}38863 \genfrac{}{}{0pt}{}{a_3}{m_3} - 0{,}25 \genfrac{}{}{0pt}{}{a_2}{m_2} + K(\mp 0{,}0383225 \genfrac{}{}{0pt}{}{m_3}{a_3} \pm 0{,}0116477 \genfrac{}{}{0pt}{}{m_2}{a_2})$$

$$+ K^2(-0{,}00180014 \genfrac{}{}{0pt}{}{a_3}{m_3} + 0{,}001257816 \genfrac{}{}{0pt}{}{a_2}{m_2} + K^3(\mp 0{,}0000112233 \genfrac{}{}{0pt}{}{m_2}{a_2}) \qquad (42$$

$$\genfrac{}{}{0pt}{}{a_{12}}{m_{12}} = -0{,}3863 \genfrac{}{}{0pt}{}{a_3}{m_3} + 0{,}25 \genfrac{}{}{0pt}{}{a_2}{m_2} + K(\pm 0{,}0377593 \genfrac{}{}{0pt}{}{m_3}{a_3} \mp 0{,}0114267 \genfrac{}{}{0pt}{}{m_2}{a_2})$$

$$+ K^2(0{,}002215007 \genfrac{}{}{0pt}{}{a_3}{m_3} - 0{,}00129083 \genfrac{}{}{0pt}{}{a_2}{m_2})$$

$$+ K^3(\mp 0{,}00000801667 \genfrac{}{}{0pt}{}{m_3}{a} \pm 0{,}0000248517 \genfrac{}{}{0pt}{}{m_2}{a_2}) \qquad (43$$

These expressions show clearly the influence of inertia forces which is represented by the terms dependent on K. Indeed in the expressions for the coefficients the first two terms which are independent of K represent exactly the before mentioned solution for zero inertia terms. One confirms this easily by replacing the variable r by y and by developing in power series of y. One notices that the series are not absolutely convergent in K and this means according to the definition of K also in the Reynoldsnumber. The first influence of inertia forces occurs in a_5, a_6, m_5, m_6 with the first power of K. Then in the three following expressions for the pairs a, m the quadratic power of K is added. The next three expressions contain also the third power and so on. Therefore this series expansion only can be used for relatively small R ynolds numbers. This is a well known peculiarity which for example also occurs when applying the method of successive approximation [8].

The expressions (35 to 43) refer to the boundary condition $u = v = 0$ at the inner cylinder where $y = 0$. This gives

$$f(0) = f'(0) = g(0) = g'(0) = 0 \tag{44}$$

what means, that

$$a_0 = a_1 = m_0 = m_1 = 0 \tag{45}$$

It may be added, that for arbitrary boundary conditions at $y = 0$ the expressions (35 to 43) contain the following additional terms

$$\begin{matrix} a_4 \\ m_4 \end{matrix} = \ldots -0,125 \begin{matrix} (a_1 - a_0) \\ (m_1 - m_0) \end{matrix}$$

$$\begin{matrix} a_5 \\ m_5 \end{matrix} = \ldots +0,15 \begin{matrix} (a_1 - a_0) \\ (m_1 - m_0) \end{matrix} + K\left(\pm 0,016 \begin{matrix} m_1 \\ a_1 \end{matrix} \mp 0,016 \begin{matrix} m_0 \\ a_0 \end{matrix}\right)$$

$$\begin{matrix} a_6 \\ m_6 \end{matrix} = \ldots -0,1625 \begin{matrix} (a_1 - a_0) \\ (m_1 - m_0) \end{matrix} + K\left(\mp 0,03 \begin{matrix} m_1 \\ a_1 \end{matrix} \pm 0,025 \begin{matrix} m_0 \\ a_0 \end{matrix}\right)$$

$$\genfrac{}{}{0pt}{}{a_7}{m_7} = \ldots + 0{,}169643 \genfrac{}{}{0pt}{}{(a_1 - a_o)}{(m_1 - m_o)} + K\left(\pm 0{,}0428571 \genfrac{}{}{0pt}{}{m_1}{a_1}\right.$$

$$\left. \mp 0{,}02619047 \genfrac{}{}{0pt}{}{m_o}{a_o}\right)$$

$$\genfrac{}{}{0pt}{}{a_8}{m_8} = \ldots - 0{,}1741071 \genfrac{}{}{0pt}{}{(a_1 - a_o)}{(m_1 - m_o)} + K\left(\mp 0{,}0498512 \genfrac{}{}{0pt}{}{m_1}{a_1}\right.$$

$$\left. \pm 0{,}0261905 \genfrac{}{}{0pt}{}{m_o}{a_o}\right) + K^2\left(-0{,}000396825 \genfrac{}{}{0pt}{}{a_1}{m_1} + 0{,}000396825 \genfrac{}{}{0pt}{}{a_o}{m_o}\right)$$

$$\genfrac{}{}{0pt}{}{a_9}{m_9} = \ldots + 0{,}177083 \genfrac{}{}{0pt}{}{(a_1 - a_o)}{(m_1 - m_o)} + K\left(\pm 0{,}0551587 \genfrac{}{}{0pt}{}{m_1}{a_1} \mp 0{,}0258433 \genfrac{}{}{0pt}{}{m_o}{a_o}\right.$$

$$\left. + K^2(0{,}00102513 \genfrac{}{}{0pt}{}{a_1}{m_1} - 0{,}0008597883 \genfrac{}{}{0pt}{}{a_o}{m_o}\right)$$

$$\genfrac{}{}{0pt}{}{a_{10}}{m_{10}} = \ldots -0{,}17916 \genfrac{}{}{0pt}{}{(a_1 - a_o)}{(m_1 - m_o)} + K\left(\mp 0{,}0592808 \genfrac{}{}{0pt}{}{m_1}{a_1} \pm 0{,}0254117 \genfrac{}{}{0pt}{}{m_o}{a_o}\right)$$

$$+ K^2\left(- 0{,}00167659 \genfrac{}{}{0pt}{}{a_1}{m_1} + 0{,}00123677 \genfrac{}{}{0pt}{}{a_o}{m_o}\right)$$

$$\genfrac{}{}{0pt}{}{a_{11}}{m_{11}} = \ldots + 0{,}180681 \genfrac{}{}{0pt}{}{(a_1 - a_o)}{(m_1 - m_o)} + K\left(\pm 0{,}0625446 \genfrac{}{}{0pt}{}{m_1}{a_1}\right.$$

$$\left. \mp 0{,}0249851 \genfrac{}{}{0pt}{}{m_o}{a_o}\right) + K^2\left(0{,}00229738 \genfrac{}{}{0pt}{}{a_1}{m_1} - 0{,}00152898 \genfrac{}{}{0pt}{}{a_o}{m_o}\right)$$

$$+ K^3\left(\mp 0{,}00000561167 \genfrac{}{}{0pt}{}{m_1}{a_1} \pm 0{,}00000561167 \genfrac{}{}{0pt}{}{m_o}{a_o}\right)$$

$$\genfrac{}{}{0pt}{}{a_{12}}{m_{12}} = \ldots -0{,}18 \genfrac{}{}{0pt}{}{(a_1 - a_o)}{(m_1 - m_o)} + K\left(\mp 0{,}0651729 \genfrac{}{}{0pt}{}{m_1}{a_1} \pm 0{,}0245930 \genfrac{}{}{0pt}{}{m_o}{a_o}\right)$$

$$+ K^2\left(- 0{,}00286902 \frac{a_1}{m_1} + 0{,}00175295 \frac{a_0}{m_0} \right)$$

$$+ K^3\left(\pm 0{,}0000184384 \frac{m_1}{a_1} \mp 0{,}0000164342 \frac{m_0}{a_0} \right)$$

The first boundary condition in (8) for the outer boundary $y = \delta$ is satisfied by the mean flow whilst the second one in (8) has to be satisfied by the secondary flow. Hence the boundary conditions for the secondary flow are

$$u = 0\,, \qquad v = -U^* \frac{\varepsilon}{\delta + 1} \sin\varphi \tag{46}$$

Introducing (33,34) one obtains

$$f'(\delta) = g'(\delta) = 0\,, \quad g(\delta) = 0\,, \quad f(\delta) = U^* \tag{47}$$

The boundary conditions (44,47) give eight equations to determine the eight constants a_0, a_1, a_2, a_3, m_0, m_1, m_2, m_3. As mentioned above the conditions (44) yield $a_0 = a_1 = m_0 = m_1 = 0$. So that the four equations (47) still have to be solved. This work was carried through numerically with the aid of computers.

7.) Numerical calculations.

The numerical quantities inserted for δ_0, ϱ (10) are

$$\delta_0 = 0{,}2 \quad , \quad \varrho = 1{,}2$$

Due to the before mentioned semiconvergence of the seri expansion there exist certain limits for the Reynolds numb beyond of which the convergence is not any more satisfactory. With the assumed value $\delta_0 = 0{,}2$ the limit is $K = 10^4$. With (25) this corresponds to a Reynolds number

$$Re = \frac{U^* r_i}{\nu} = 4 \cdot 10^3$$

The calculations were performed for two values of K, respectively Re. The boundary conditions (47) give the following expressions

1.) $K = 3 \cdot 10^3$, $Re = 1{,}32 \cdot 10^3$

$$f(\delta) = 0{,}026011\, a_2 + 0{,}005206\, a_3 + 0{,}025806\, m_2 + 0{,}004302\, m_3 = U^*$$

$$g(\delta) = -0{,}025806\, a_2 - 0{,}004302\, a_3 + 0{,}026011\, m_2 + 0{,}005206\, m_3 = 0$$

$$f'(\delta) = -0{,}15057\, a_2 + 0{,}01557\, a_3 + 0{,}54345\, m_2 + 0{,}09745\, m_3 = 0$$

$$g'(\delta) = -0{,}54345\, a_2 - 0{,}09745\, a_3 - 0{,}15057\, m_2 + 0{,}01557\, m_3 = 0$$

The solution is

$$a_2 = 56{,}03833\, U^* \qquad m_2 = -54{,}48119\, U^*$$
$$a_3 = -161{,}82191\, U^* \qquad m_3 = 416{,}26530\, U^* \tag{48}$$

2.) $K = 10^4$, $Re = 4{,}4 \cdot 10^3$

$$f(\delta) = -0{,}118805\, a_2 - 0{,}016182\, a_3 - 0{,}030515\, m_2 - 0{,}015541\, m_3 = U^*$$

$$g(\delta) = 0{,}030515\, a_2 + 0{,}015541\, a_3 - 0{,}118805\, m_2 - 0{,}016182\, m_3 = 0$$

$$f'(\delta) = 0{,}578207\, a_2 + 0{,}090828\, a_3 + 0{,}413482\, m_2 + 0{,}146802\, m_3 = 0$$

$$g'(\delta) = -0{,}413482\, a_2 - 0{,}146802\, a_3 + 0{,}578207\, m_2 + 0{,}090828\, m_3 = 0$$

The solution is

$$a_2 = -30{,}40309\, U^* \qquad m_2 = 2{,}63489\, U^*$$
$$a_3 = 119{,}69097\, U^* \qquad m_3 = 38{,}27269\, U^* \tag{49}$$

Now the excentricity can be evaluated for which separation will occur. It is to be expected that separation first occurs at the inner boundary $y = 0$. The beginning of separation will be characterized by the zero value of the derivative of the total circumferential velocity in radial direction. This means

$$U'(0) + u'(0) = 0 \tag{50}$$

(23) gives

$$U(y) = \frac{\delta+1}{\delta(\delta+2)}\, U^* \left(y + 1 - \frac{1}{1+y}\right)$$

$$U'(y) = \frac{\delta+1}{\delta(\delta+2)}\, U^* \left(1 + \frac{1}{(1+y)^2}\right)$$ (51a

Inserting δ = o,2 one obtains

$$U'(y) = 2{,}\overline{72}\ldots\, U^* \left(1 + \frac{1}{(1+y)^2}\right)$$

$$U'(0) = 5{,}\overline{45}\ldots\, U^*$$ (51b

By differentiation of (33) one derives

$$u'(y) = -\varepsilon\{f''(y)\cos\varphi + g''(y)\sin\varphi\}$$

$$= -\varepsilon\{[2a_2 + 6a_3y + \ldots]\cos\varphi + [2m_2 + 6m_3y + \ldots]\sin\varphi\}$$

$$u'(0) = -\varepsilon\{2a_2\cos\varphi + 2m_2\sin\varphi\}$$

(52

Introducing (51,52) in (50) one obtains

$$5{,}45\, U^* - \varepsilon\,(2a_2\cos\varphi + 2m_2\sin\varphi) = 0$$ (53

The coefficients were calculated before for two values of K in (47,48). Therefore (53) determines the critical excentricity for certain angles φ. The smallest excentricity obtained should be regarded as the critical value. It is sufficient to calculate the excentricity in each case for two angles.

1.) $K = 3 \cdot 10^3$

$\varphi = 0^0$ $5{,}45\, U^* - 2\varepsilon\, a_2 = 0$

$\varepsilon = 0{,}04867$, $e = 1{,}01392$ mm

$\varphi = 270^0$ $5{,}45\, U^* + 2\varepsilon m_2 = 0$

$\varepsilon = 0{,}05006$ $e = 1{,}042$ o mm

2.) $K = 10^4$

$\varphi = 180^0$ $\varepsilon = 0{,}08970$, $e = 1{,}86875$ mm

$\varphi = 90^0$ $\varepsilon = 1{,}03506$. $e = 21{,}56375$ mm

8.) Comparison with experiments.

To prove the theory experiments were performed with a Couette aparatus the cylinders of which were excentric. The dimensions of the apparatus were the following

r_o = 21 mm $\qquad\qquad$ ϱ = 1,19

r_1 = 25 mm $\qquad\qquad$ δ = o,19

e = o,5; 1; 2; 2,5; 3 mm $\qquad\qquad$ ε = o,o476 (e = 1mm)

As shown in fig 3 no ball bearings were used for the support of the inner cylinder. The occurance of separation was observed visualy with a technique described earlier [9]. With the excentricity e = 1,5 mm there was found a critical Reynolds number for separation Re = $U^* r_0 / \nu$ = 1,69 . 1o^4 [5], These experiments were now extended to the before mentioned excentricities. With e = o,5 mm no separation could be observed up to the Reynolds number 4 . 1o^5. The critical Reynolds numbers which were observed are plotted in fig 4. As this figure shows there exists a definite dependence on the excentricity. This confirms the supposition that turbulence is generated by separation.

The calculation for Re = 1,3 . 1o^3 gave separation with ε = o,o48. This excentricity was realized in the experiments and as fig 4 shows the corresponding Reynolds number for separation is 2,1 . 1o^4. One sees that the theoretical value is 16 times to small. This seems to be a satisfactory agreement for a first order approximation.

It may be mentioned that earlier calculations of the separation at a corrugated plate showed a sensitive influence of the corrugation [7]. Now according to the experimental comparison this sensitivity seems to be more influenced by the degree of approximation than by physical effects. This makes an evaluation of the inertia forces necessary.

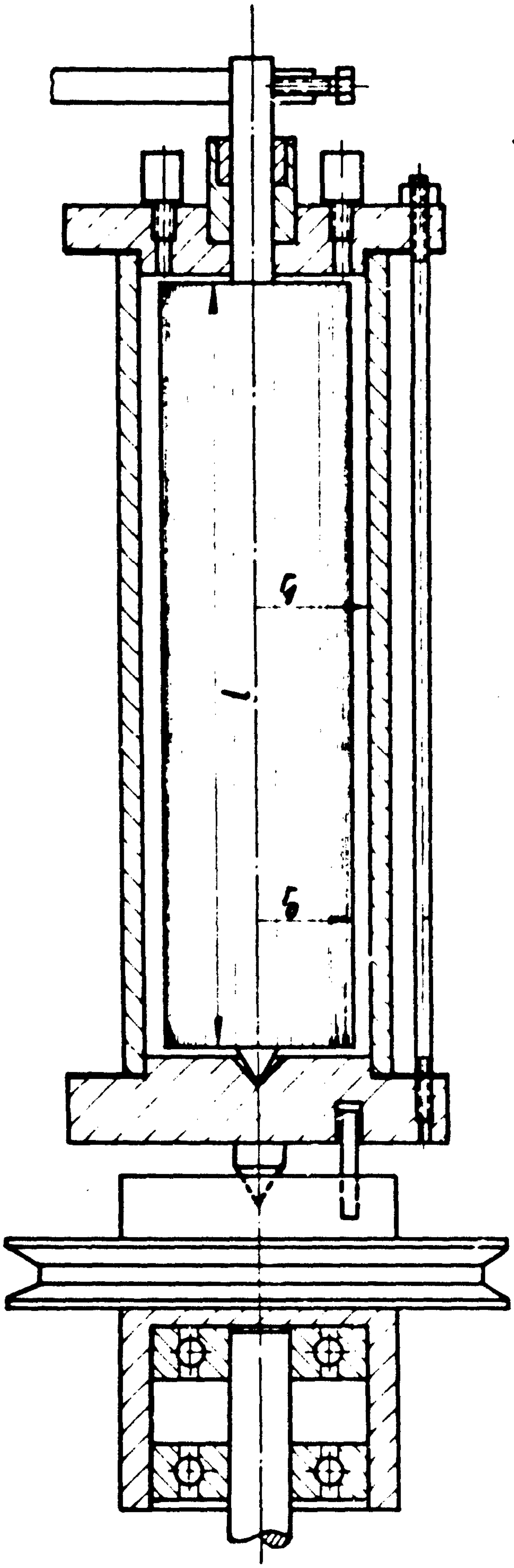
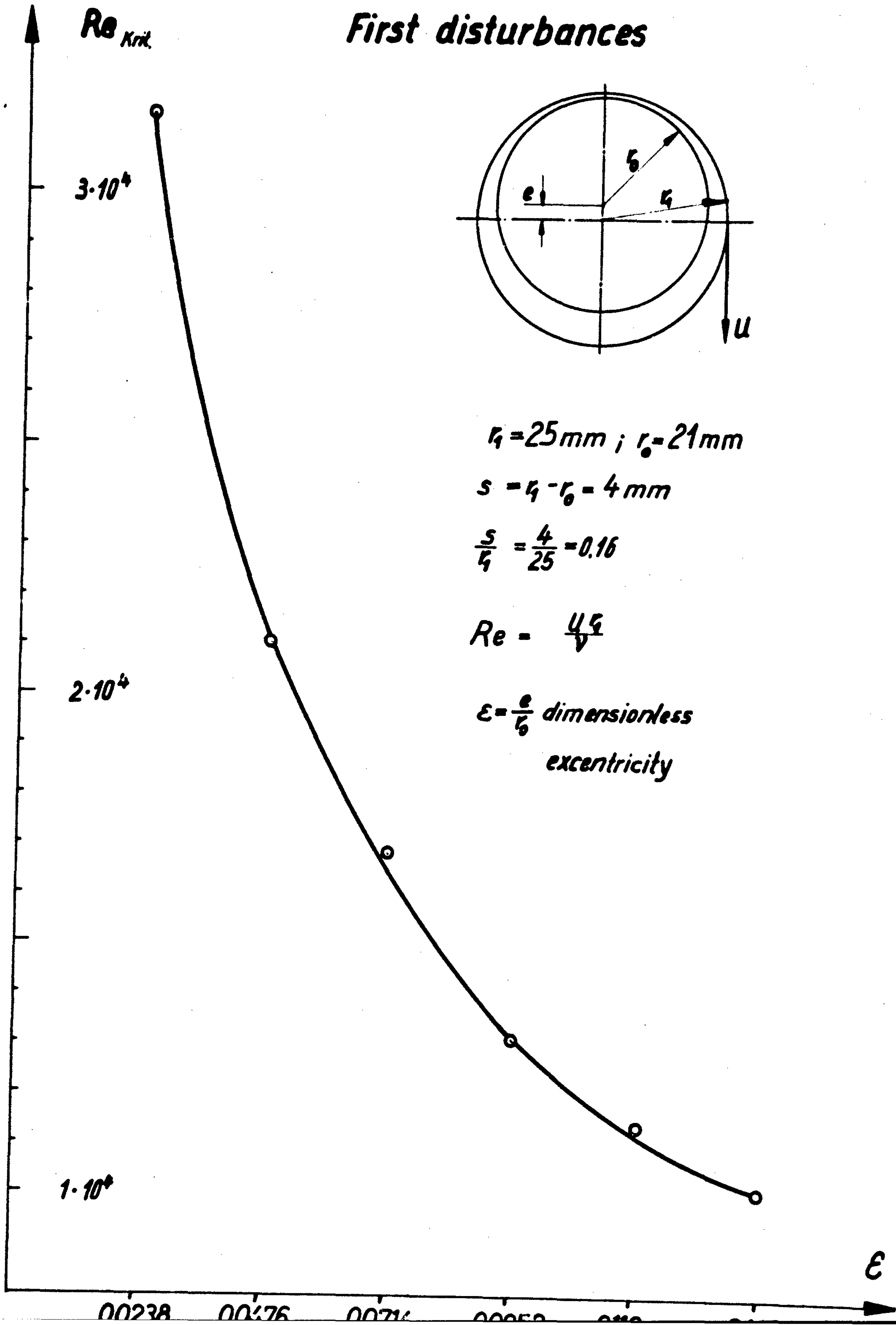


Fig. 3



Re Krit.
First disturbances
3·10^4
2·10^4
1·10^4
e
r0
r1
u
r1 = 25mm ; r0 = 21mm
s = r1 - r0 = 4mm
s/r1 = 4/25 = 0,16
Re = u r1/ν
ε = e/r0 dimensionless
excentricity
ε
00238

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